Electronic Design

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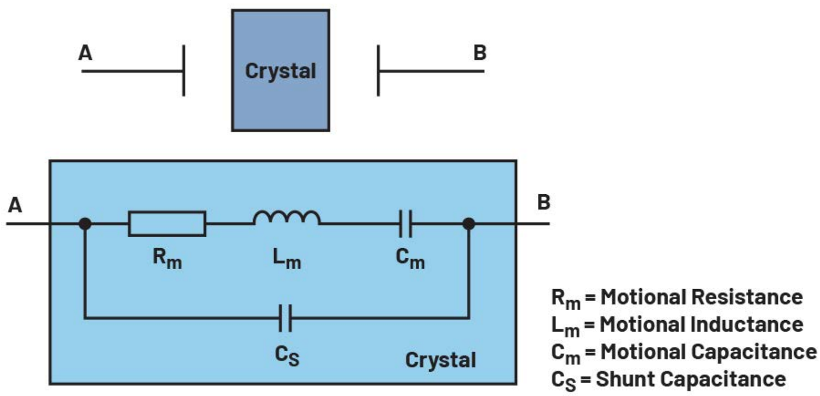
The Crystal-Clear Truth About Critical Transconductance

This article delves into critical transconductance for crystal oscillators with a Pierce architecture, covering its physical meaning and providing exact step-by-step derivations.

Crystal oscillators are essential for precise timing in electronic systems, and most engineers are familiar with their basic principles. Formulas for critical and optimum transconductance are widely used in design, yet their origins often remain opaque.

This article aims to uncover that reasoning, starting from fundamental concepts such as the crystal's impedance and the Pierce oscillator topology, encompassing oscillator feedback strategies, and proceeding through the derivations step by step. It will demonstrate how these formulas emerge from circuit analysis, allowing the reader to not only understand the formulas, but also know exactly where they come from and why they matter.



1. Quartz-crystal equivalent electrical model.

Setting the Frequency

A crystal is a small piece of quartz that exploits the piezoelectric effect: An applied electric field induces mechanical vibration, and mechanical stress generates an electric response. It can be modeled electrically as seen in *Figure 1*.

The crystal itself can exhibit series resonance at the frequency:

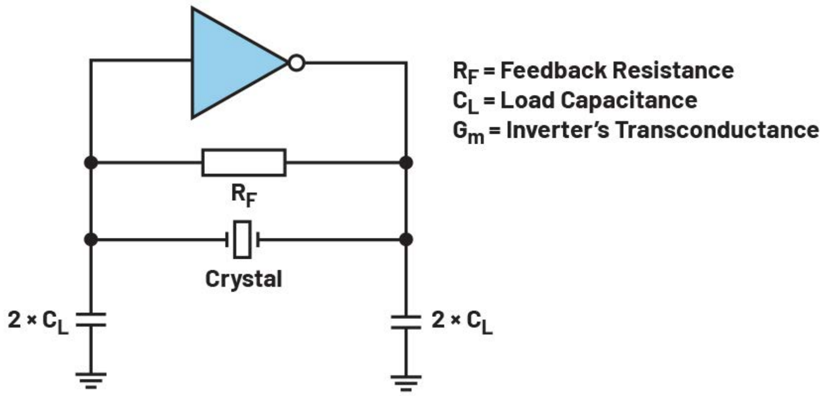
$$f_S = \frac{1}{2 \times \pi \times \sqrt{L_m \times C_m}} \quad (1)$$

It also exhibits antiresonance (when in parallel mode) at the frequency:

$$f_P = \frac{1}{2 \times \pi \times \sqrt{L_m \times \left(\frac{C_m \times C_S}{C_m + C_S}\right)}} = f_S \times \sqrt{1 + \left(\frac{C_m}{C_S}\right)} \quad (2)$$

When an extra capacitor C_L is added in parallel with the crystal, the crystal is loaded, and the oscillating frequency becomes:

2. This is a Pierce oscillator circuit.



$$f_L = \frac{1}{2\times\pi\times\sqrt{L_m \times \left(\frac{C_m \times \left(C_S + \frac{C_L}{2}\right)}{C_m + \left(C_S + \frac{C_L}{2}\right)}\right)}} \quad (3)$$

$$= f_S \times \sqrt{1 + \frac{C_m}{C_S + \frac{C_L}{2}}}$$

Typically, $C_m \ll C_S$, which means the series and parallel resonance frequencies (f_S and f_P) are very close. Interestingly, the loaded oscillation frequency always falls between f_S and f_P.

A widely used topology for crystal oscillators is the Pierce configuration *(Fig. 2)*.

The resistor R_F biases the inverter into its linear region, setting the DC operating point without significantly affecting the AC loop behavior. For AC analysis, its influence is negligible and will be omitted. Throughout this article, the inverter is modeled as an ideal transconductance (G_m).

Oscillator Families: Positive Feedback vs. Negative Resistance

Harmonic oscillators fall into two main categories: positive- feedback oscillators and negative-resistance oscillators.

Positive-feedback oscillators operate by feeding a portion of the output signal back to the input with the correct amplitude and phase to satisfy the Barkhausen stability criterion:

$$|A \times B| = 1 \text{ and } \varphi\,(A,B) \equiv 0 \ (mod\ 2\times\pi) \quad (4)$$

where A is the gain and B is the feedback transfer function. This approach is typically used when the gain and feedback paths are explicitly defined, as in Wien bridge or ring oscillators.

Negative- resistance oscillators, however, aim to cancel out the resistive (lossy) elements in the circuit, allowing energy to transfer freely between reactive components. The condition for oscillation becomes:

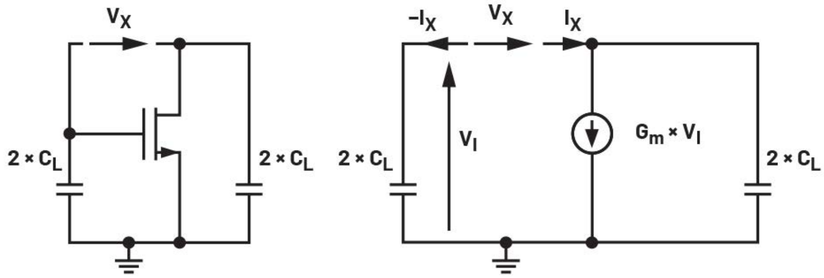
$$\sum_n (R_n) + R_{neg} = \sum_n (jX_n) = 0 \quad (5)$$

This perspective is more intuitive when a resonator is present, such as an LC tank or a crystal. Common examples include Hartley, Colpitts, and Clapp oscillators.

Though conceptually different and relying on distinct analytical viewpoints, both approaches describe the same physical phenomenon.

Why Critical G_m Matters in Crystal Oscillators

Among the key design parameters in crystal oscillator circuits, the transconductance plays a central role. If G_m is too low, the oscillator simply won't start — a failure that can be both subtle and frustrating. To ensure a reliable startup, G_m must exceed a threshold known as the critical G_m.



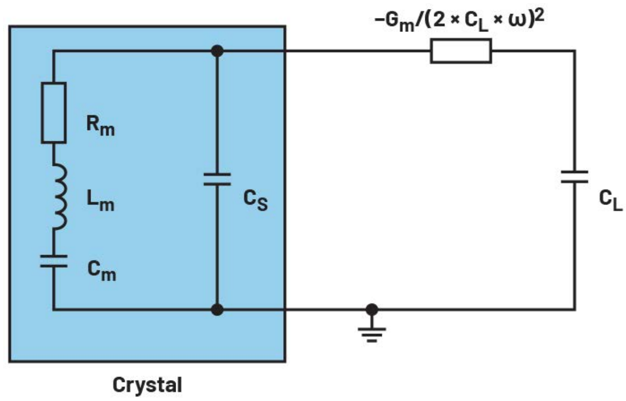
3. Generation of negative impedance.

The Thumb Rule Trap

A simplified expression sometimes encountered for estimating critical transconductance is:

$$G_mCRIT = \frac{1}{R_m} \tag{6}$$

This approximation loosely stems from the Barkhausen criterion and resonance behavior. At series resonance, the crystal's impedance simplifies to $Z \approx R_m$ (neglecting C_S), and the loop gain is approximated as $A \times B = G_m \times R_m > 1$. However, this model doesn't reflect the Pierce oscillator topology and produces results that are not only incorrect, but also directly contrary to the intended behavior. As shown later, an increase in R_m actually demands a higher G_m.



4. Shown is an equivalent circuit.

Generating a Negative Resistance

As noted earlier, because the circuit includes a resonator, the negative-resistance approach is the more suitable method of analysis. A simple way to create a negative resistance is to look at the input impedance of the topology *(Fig. 3)*.

Calculate the dynamic impedance seen at V_X to prove it's of the form:

$$Z_X(j\omega) = \frac{-G_m}{(2 \times C_L \times \omega)^2} + \frac{1}{j \times C_L \times \omega} \tag{7}$$

which created a negative resistance equal to:

$$Re\{Z_X(j\omega)\} = \frac{-G_m}{(2 \times C_L \times \omega)^2} \tag{8}$$

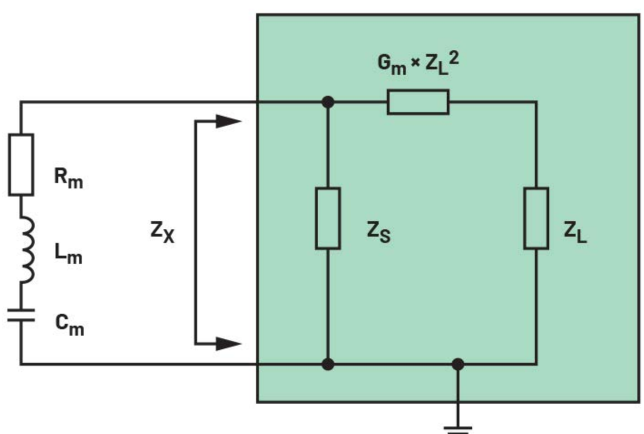
The AC circuit seen by the crystal is equivalent to the circuit in *Figure 4*.

Considering only the motional branch (assuming C_S to be negligible), the oscillations will be achieved for $G_m > 4 \times R_m \times (C_L \times \omega)^2$.

The Conventional Approach

A convenient way to account for the shunt capacitance is to absorb it into the impedance seen by the crystal, Z_x *(Fig. 5)*.

Using basic impedance transformation, derive:



5. Impedance seen from the crystal.

$$Z_X = \frac{G_m \times Z_S \times Z_L^2 + Z_S \times Z_L}{G_m \times Z_L^2 + Z_S + Z_L} \qquad (9)$$

The real part of that impedance can be shown as:

$$Re\{Z_X(j\omega)\} = \frac{-G_m \times 4 \times C_L^2}{(4 \times C_S \times C_L + 4 \times C_L^2)^2 \times \omega^2 + (G_m \times C_S)^2} \qquad (10)$$

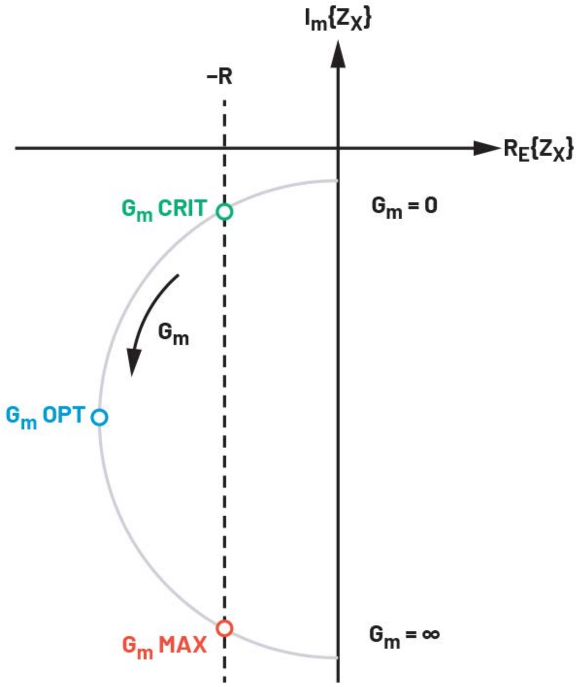
To offset the resistive losses, the negative resistance must equal the motional resistance:

$$R_m = \frac{G_m \times 4 \times C_L^2}{(4 \times C_S \times C_L + 4 \times C_L^2)^2 \times \omega^2 + (G_m \times C_S)^2} \qquad (11)$$

which leads to the following quadratic equation:

$$\rightarrow G_m^2 - \frac{4 \times G_m}{R_m} \times \left(\frac{C_L}{C_S}\right)^2 + 16\left(C_L + \frac{C_L^2}{C_S}\right)^2 \times \omega^2 = 0 \qquad (12)$$

Calculating the discriminant Δ will allow us to determine if a real solution (R) exists.



6. Impedance in the complex plane.

$$\Delta = \frac{16}{R_m^2} \times \left(\frac{C_L}{C_S}\right)^4 - 64 \times \left(C_L + \frac{C_L^2}{C_S}\right)^2 \times \omega^2 \qquad (13)$$

If the discriminant Δ is negative, the system has no real solutions. If Δ is positive, two values of G_m satisfy the equation, and oscillation is possible for any value in between. The lower bound is known as the critical transconductance G_m_CRIT, which is the minimum required for startup. The upper bound is the maximum transconductance G_m_MAX, beyond which stable oscillation can't be sustained. Exact values can be determined using a numerical solver:

$$G_m\ CRIT = \frac{\frac{4}{R_m} \times \left(\frac{C_L}{C_S}\right)^2 - \sqrt{\Delta}}{2}$$

$$G_m\ MAX = \frac{\frac{4}{R_m} \times \left(\frac{C_L}{C_S}\right)^2 + \sqrt{\Delta}}{2} \qquad (14)$$

Solving the condition:

$$\frac{\partial Re\{Z_X(j\omega)\}}{\partial G_m} = 0 \qquad (15)$$

can determine the optimum transconductance, which corresponds to the point where the negative real part of the impedance seen by the crystal is maximized. This yields the expression:

$$G_m\ OPT = 4 \times \omega \times C_L \times \left(1 + \frac{C_L}{C_S}\right)^2 \qquad (16)$$

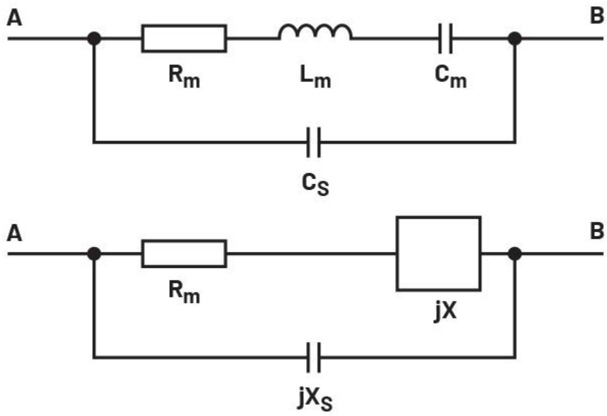
As G_m increases from 0 toward +∞, the impedance seen by the crystal evolves in the complex plane. This progression illustrates the transition from critical G_m, through optimum G_m, up to maximum G_m. Outside this range, the losses can't be compensated, and the system can't sustain oscillations *(Fig. 6)*.

The value of interest is usually G_{m}_CRIT. Assuming $G_m \times C^2 \ll 4 \times (C \times C + C^2)^2 \times \omega^2$, then the impedance Z can be approximated as:

$$R_m = -Re\{Zx(j\omega)\} \approx \frac{-G_m \times 4 \times C_L{}^2}{(4 \times C_S \times C_L + 4 \times C_L{}^2)^2 \times \omega^2} \quad (17)$$

This simplification is sometimes referred to as the small G_m approximation. The physical intuition is that R_m is small (as crystals have very high-quality factors, ranging from 10,000 to 100,000), and thus the associated losses are minimal. Consequently, the contribution of the negative impedance circuit (G_m) should be minimal. Under this assumption, the critical G_m can be expressed in a more compact form:

$$G_{mCRIT} = 4 \times R_m \times (2 \times \pi \times fs)^2 \times (C_L + C_S)^2 \quad (18)$$

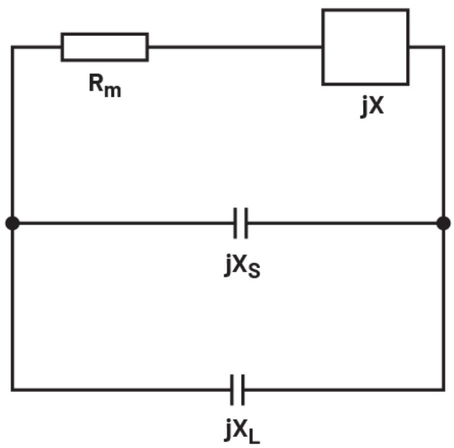


7. Here's an alternate crystal model.

A Detour that Delivers Insight

The previous method treats the crystal as a general impedance, without distinguishing between its resistive and reactive components. An alternative approach exists whereby, while slightly longer to derive, offers deeper insight and practical benefits.

To begin, simplify the motional branch of the crystal by grouping its inductive and capacitive elements into a single reactance term, X. For consistency in notation, the shunt capacitance reactance is defined as jX_S, with $X_S = -1/(C_S \times \omega)$ *(Fig. 7)*.



8. An alternate crystal model loaded.

The impedance of the crystal (between A and B) can be written as:

$$Z_{AB} = \frac{(R_m + jX) \times (jX_S)}{R_m + j(X + X_S)} \quad (19)$$

The effective resistance between nodes A and B corresponds to the real part of the impedance seen across those points:

$$R_{EQ} = Re\{Z_{AB}\} = \frac{R_m \times X_S{}^2}{R_m{}^2 + (X + X_S)^2} \quad (20)$$

With the crystal connected to the load capacitor, the circuit can now be drawn as shown in *Figure 8.*

At resonance, the reactive components cancel each other out, hence:

$$X = X_S // X_L$$

$$X = \frac{1}{\omega \times (C_L + C_S)} \quad (21)$$

Inserting this result into the previous R_{EQ} equation gives us:

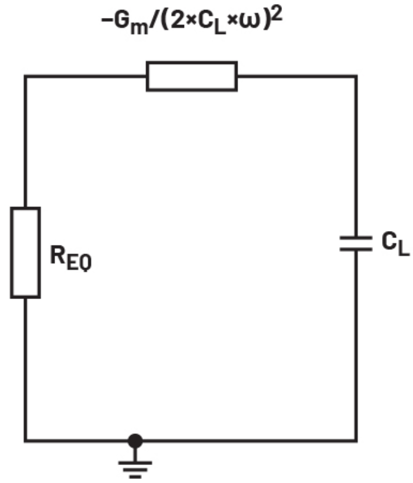
$$R_{EQ} = Re\{Z_{AB}\} = \frac{R_m \times X_S^2}{R_m^2 + \left(\frac{1}{\omega \times (C_L + C_S)} - \frac{1}{\omega \times C_S}\right)^2} \quad (22)$$

which can be simplified as:

$$R_{EQ} = \frac{R_m \times X_S^2}{R_m^2 + \frac{1}{C_S^2 \times \omega^2}\left(\frac{C_L}{C_L + C_S}\right)^2} \quad (23)$$

If the assumption:

$$R_m \ll \frac{1}{C_S \times \omega}\left(\frac{C_L}{C_L + C_S}\right) \quad (24)$$



9. Shown is another equivalent circuit.

is true, the crystal's equivalent resistance seen by the amplifier can be approximated as:

$$R_{EQ} = R_m\left(1 + \frac{C_S}{C_L}\right)^2 \quad (25)$$

This formula provides a useful link between the crystal's internal power dissipation, its motional resistance R_m, and the capacitance ratio C_S/C_L. Crystals have a maximum allowable power dissipation, beyond which long-term reliability may be compromised. The dissipated rms power can now be estimated as $P_{rms} = R_{EQ} \times I\ ms^2$ to verify that the oscillator operates within safe drive levels.

Unfortunately, the term R_{EQ} is often referred to as equivalent series resistance (ESR) and used inconsistently in oscillator design. In some contexts, ESR refers to the effective resistance seen by the amplifier (what's been called R_{EQ}). In others, it denotes the motional resistance (electrical series resistance), which represents the intrinsic losses within the crystal itself (what's been called R_m).

To add to the confusion, some vendors label R_m as ESR, while certain documents use ESR to describe R_m in the context of critical G_m calculations. Typically, C_S is much smaller than C_L, making R_m and R_{EQ} numerically close. This may explain why the distinction often goes unnoticed.

Hopefully, this clarification helps the reader differentiate between the two and understand which value is appropriate in each context: calculating startup conditions, resonant frequency, or estimating drive level.

Returning to the equivalent circuit, the oscillator can now be represented as shown in *Figure 9*.

Once again, to sustain oscillation, the negative resistance presented by the amplifier must cancel out the equivalent resistance seen from the crystal:

$$\frac{G_{mCRIT}}{(2 \times C_L \times \omega)^2} = R_m\left(1 + \frac{C_S}{C_L}\right)^2 \quad (26)$$

Rearranging Equation 26 gives the same expression previously derived for the critical transconductance:

$$G_{mCRIT} = 4 \times R_m \times (2 \times \pi \times f)^2 \times (C_L + C_S)^2 \quad (27)$$

The Power of Four

Ultimately, the critical G_m represents the precise mathematical threshold required for sustained oscillation. It ensures that the energy lost in the crystal is exactly compensated by the amplifier.

However, to guarantee a reliable startup under real-world conditions, designers typically provide excess G_m. This accounts for variations in R_m, drive-level dependencies, and startup dynamics. A safety factor of 4 is commonly applied in practice to ensure robust performance across temperature, aging, and process variations.

Conclusion

This article demonstrates why critical G_m is essential for crystal oscillator startup and stability. We explored the dependencies with R_m, C_S, and C_L, highlighted the common pitfall of relying solely on $1/R_m$, and presented two complementary approaches to solving the problem. Each method offers unique insights: one provides a complete view of the operating range, while the other reveals a key relationship with drive level.

Understanding these principles equips designers to move beyond rules of thumb and apply precise calculations that ensure reliable startup and robust performance under real-world conditions.

Jerome Rollet is a staff analog design engineer in the Data Center and Energy Group at Analog Devices. Prior to joining ADI in 2014, he was a member of technical staff at Maxim Integrated. Jerome has 17 years of expertise in power management and mixed-signal circuit design across consumer, industrial, and automotive applications. He holds a Master of Science in electronic engineering from the Grenoble Institute of Technology and a Master of Research in microelectronics and nanoelectronics from Grenoble Alpes University.

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